Dr. Robin Rehagen, LPC Physics

Understanding Experimental Error

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1. Overview

It is essential in lab class to understand the limitations of your experiments. Scientists keep track of their experimental limitations in several ways:

- **Know the assumptions made when using a certain physical model.** For example, we often assume that we can ignore air resistance when using kinematics equations.
- **Be aware of bias: instances in which your sample of data may be inherently different than the population you wish to study.** For example, say I want to study the average height of a human. If I chose to only ask professional basketball players to give their heights and no one else, then my final answer will be biased to very high values because basketball players do not consist of a representative sample of the world population.
- **Know that every measurement you make has some uncertainty associated with it.** This uncertainty may come from limitations of your measuring device (like the size of the tick marks on a ruler) or the random variations that are associated with everyday life (if you want to measure human body temperature, every healthy hum will have a slightly different temperature).

A more detailed list of the types of error that may come up in your experiments may be found in Section 5. As you write the discussion section of your lab report, you may use this list to help you evaluate the types of error present in your experiment.

Note that the words *error* and *uncertainty* are equivalent, and completely interchangeable.

2. Accuracy vs. Precision

We will start with some definitions:

- **Accuracy indicates how close your experiment is to the "right answer".** If you knew in advance that your internal body temperature was 98.4°F, then you would say a thermometer is accurate if it could reproduce that known value.
- **Precision indicates how well your experiment can reproduce the same result.** If you took your temperature ten times, and the thermometer always read 98.9° on each measurement, it would be very precise – but not accurate.

It is important that our experiments are **both** precise and accurate. Often, accuracy is hard to determine in real experiments – we don't always know what the answer "should be". In fact, in scientific research we almost **never** know what the answer "should be". Therefore, when scientists talk about "uncertainty" or "error" on a measured value, they are almost always talking about the precision of their measurement, not the accuracy.

3. Measuring Accuracy

In order to measure accuracy, we must know the "answer" to our scientific question ahead of time. Sometimes this will be relevant in lab class. For instance, if I am trying to measure *g*, the acceleration due to gravity, I know from previous experiments that the average value of *g* has been found to me 9.8 m/s^2. Therefore, I can measure the accuracy of my experiment. Let's say that I measured g = 9.7 m/s^2. In order to determine how accurate my experiment was, I can calculate a **percent difference** from the accepted vale of *g*:

$$\text{Percent Difference} = \left|\frac{\text{My Value} - \text{True Value}}{\text{True Value}}\right| = \left|\frac{9.7\,\frac{\text{m}}{\text{s}^2} - 9.8\,\frac{\text{m}}{\text{s}^2}}{9.8\,\frac{\text{m}}{\text{s}^2}}\right| = 0.01$$

For this experiment, I would say that my percent difference from the accepted value for *g* is 1%. This statement gives the accuracy of my experiment.

In lab, it is not always possible to determine a percent difference. If you have no "known value" to compare your answer to, you cannot calculate a percent difference. If you are able to calculate it, then you should use it to test the accuracy of your experiment. If you find that your percent difference is more than 10%, there is likely something wrong with your experiment and you should figure out what the problem is and take new data.

4. Measuring Precision

Precision is measured using two different methods, depending on the type of measurement you are making. These methods are described in Sections 4.1 and 4.2. There are special circumstances when more complicated methods are necessary, such as finding the errors in parameters of a fit (discussed in Section 4.3) or finding the error in a trigonometric function (discussed in Section 4.5). Depending on the details of your experiment, you may need to propagate your measurement errors as described in Section 4.4. At the end of your experiment, always report your results with their associated error, and calculate a percent error (as described in Section 4.6).

4.1 Instrumental Resolution

One type of measurement uncertainty is due to **instrumental resolution**. If you are trying to measure the length of a piece of string, the precision of your length measurement depends on what type of ruler you use to measure the length. The smaller the tick marks on your ruler, the more precise your measurement. Similarly, if you are weighing an object on a digital scale, your precision is limited by the number of decimal places that the scale gives you. More decimal places on the scale will give you a more precise measurement. Thus, the rules for measuring instrumental precision are:

- If you are taking a measurement with a device labeled with tick marks, the error in your measurement is half the size of a tick mark.

  If you measure a length of string to be 15 cm long with a ruler where the smallest tick mark is 1 mm. Since half a millimeter is 0.05 cm, the value quoted in your lab report should be: 15.0 $\pm$0.05 cm.

- If you are taking a measurement with a digital device, the error in your measurement is half the size of the 10ths place of the smallest digit.

  If you measure the mass of an object to be 6.26 g, then the tenths place of the smallest digit is 0.01 g, so half of this value would be 0.005 g. Thus, the measurement you would quote in your lab report is 6.26 $\pm$0.005 g.

4.2 Random Error

The second type of measurement uncertainty is due to random error. The errors described in Section 4.1 do no fluctuate randomly. If I weighed the object on the digital scale ten times, I should get 6.26 g each time I put it on the scale. However, other quantities in lab might fluctuate randomly with every measurement you take. The following page shows some data taken by a student who wants to measure the acceleration due to gravity.

	Acceleration (m/s^2) (± 0.01 m/s^2)
Trial 1	9.81
Trial 2	9.70
Trial 3	9.84
Trial 4	10.05
Trial 5	9.64
Trial 6	9.77
Trial 7	9.91
Trial 8	9.93
Trial 9	9.60
Trial 10	9.91
Mean	**9.82 $\pm$ 0.04 m/s^2**

Table 1 – Here we show the acceleration of the falling ball as calculated in each of the eight trials, along with the mean acceleration of all trials. The error in individual acceleration measurements is 0.01 m/s^2. Our final result is a measurement of $g = 9.82 \pm 0.04$ m/s^2.

This student measured g, the acceleration due to gravity, ten separate times. Even though we know that the true value of g does not change from trial to trial, the student's measurements do. The student correctly quotes the instrumental uncertainty on his measurement device to be $\pm$ 0.01 m/s^2. However, we can see that the actual numbers vary much more widely that $\pm$ 0.01 m/s^2. How should he quantify his uncertainty?

The first thing to note is that in cases where your measurements fluctuate randomly, you must take multiple trials (10 minimum) in order to get a good idea of the amount of fluctuation. Once you have taken multiple trials, you should then find the **mean** of your measurements. The mean tells you your "best guess" of what the correct answer is, and this is the final numerical result that you should quote in your lab report.

There is a simple formula to determine the error on the mean value

$$\text{Error on the Mean} = \frac{\sigma}{\sqrt{N}}$$

where σ is the standard deviation of your data and N is the total number of data points (10, in this case). Standard deviation σ is defined as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}}$$

where $\bar{x}$ is the mean of your measurements and each x_i represents an individual measurement. Don't worry about the details of this formula, you can quickly use the Excel function =STDEV() to calculate σ for you. But it is worth pausing now to take a look at what Excel is actually calculating when you use =STDEV(). The term $(x_i - \bar{x})$ is the *difference between an individual measurement and the mean.* We then square that term so positive and negative deviations from the mean are treated the same. We add up the $(x_i - \bar{x})^2$ deviations for each individual measurement, and then divide by the total number of measurements to get an "average" deviation. (Note that we divide by $N - 1$ instead of N for complicated reasons that are best discussed in a statistics class.) Finally, we take the square root to undo the fact that we squared the $(x_i - \bar{x})$ term. **Thus, the standard deviation is a measurement of the average amount that the data deviates from the mean value.**

To recap: If your data fluctuates between different trials, take the mean of the data and calculate the error on the mean. For data that fluctuates, the error on the mean gives you a better estimate of your precision than the instrumental errors on individual data points. For fluctuating data, the error on the mean is almost always larger than the instrumental error, and is thus a more conservative way to measure your experimental uncertainty.

Note: If you take significantly fewer than 10 trials, the error on the mean becomes less meaningful. Ideally, you should always perform at least 10 trials in any experiment to quantify your random error. If for some reason you only took a very small data sample (fewer than 10 trials), then a "quick and dirty" way to estimate your error is to use the half-range formula:

$$e = \frac{(\text{maximum data value}) - (\text{minimum data value})}{2}$$

The numerator represents the full range of your data. When you divide by 2, you get "half the range". This will give you an idea of the uncertainty in your measurements; however, running 10 or more trials and calculating the error on the mean is much more rigorous.

4.3 Errors in Parameters of Fitted Functions

If you have created a scatter plot with your data and are fitting a function to it, the parameters of the fitted function will all have errors. In some cases, LoggerPro is able to calculate the errors in the fit parameters. Please consult your instructor during lab to verify whether or not LoggerPro is properly estimating fit parameter errors.

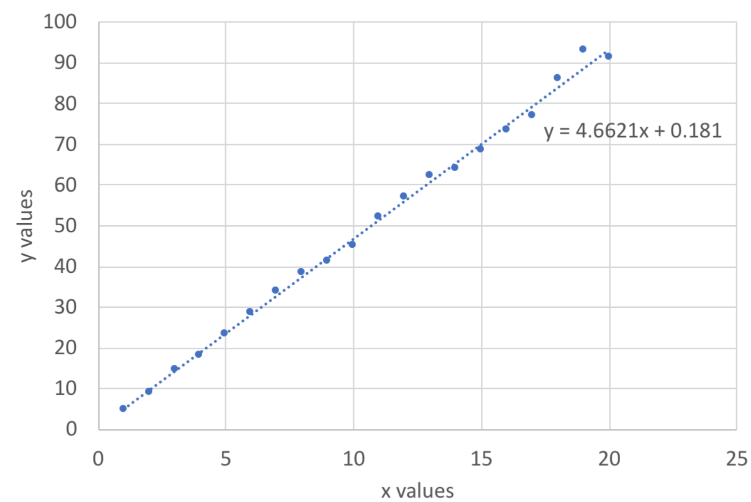
In the cases where LoggerPro does not accurately estimate fit parameter errors, you can use Excel to calculate the errors for you. Excel can only do this for a *linear function* (i.e., a straight-line function). The process of using Excel to calculate errors in a linear fit is described below.

The Excel function LINEST ("line statistics") is able to calculate the errors in the *slope* and *y-intercept* of a linear function of the form $y = mx + b$. To do so, follow the directions below:

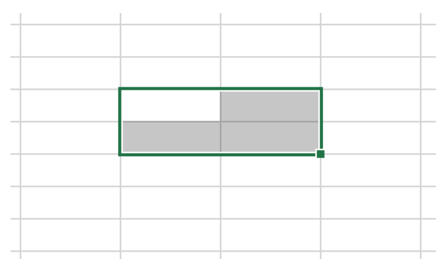
x-vals	y-vals
1	4.82873
2	9.12395
3	14.7211
4	18.1918
5	23.4051
6	28.5312
7	33.8257

1. Organize your data into a column of x-values and y-values.

2. Create a scatter plot of your data and fit a linear trendline. Display the equation for the trendline fit on your graph so that you know what the slope and intercept are.



3. Select a 2x2 box of empty cells. When you make the selection, start with the box on the upper left and move down to the box on the lower right.



4. In the selection box, type: =LINEST([select y-values],[select x-values],1,1)

5. Don't hit return! Instead, hit CONTROL+SHIFT+RETURN.

6. The empty 2x2 square should populate with numbers. The top line of numbers shows your *slope* and *y-intercept*. These 2 numbers should match the equation for your fit. The bottom line shows *error in slope* and *error in y-intercept.*

	slope	**intercept**
values:	4.66212	0.18099
errors:	0.06388	0.76518

4.4 Error Propagation

Often in a laboratory setting, we have to take measurements of many different variables in order to calculate the quantity which will give us the answer to our scientific question. How do the errors in all of our separate variables combine together? The error propagation equation tells us how these errors combine into the error in our final answer.

For example, say you wanted to calculate the time it would take for a ball to fall a very small distance (for example, the length of your index finger). Using a stopwatch to measure the time directly would be very inaccurate because of poor human reaction time, so you decide to use kinematic equations to model an equation for the time of fall. You find that the time of fall should be

$$t = \sqrt{\frac{2h}{g}}$$

You then choose to measure h and g and use those measurements to determine the time t. You measure the length your index finger, because you know that this is the height h you are dropping the ball. You measure $h = 5.2 \pm 0.05$ cm, or in SI units: $h = 0.052 \pm 0.0005$ m. You chose to use an **instrumental uncertainty** on this measurement because the length of your finger does not fluctuate randomly. Thus the uncertainty of 0.05 cm is half of the smallest tick mark on your ruler.

Next, you measure g. We will assume that you took the data shown earlier in Table 1, and you have calculated a mean value of g to be 9.82 ± 0.04 m/s^2. The uncertainty of 0.04 m/s^2 is the **error on your mean**, which is the correct way to calculate uncertainties for fluctuating data.

Now – how do you calculate t? And what is the uncertainty in t? To calculate t, you simply plug in your measured values into the equation:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(0.052\ \mathrm{m})}{9.82\,\frac{\mathrm{m}}{\mathrm{s}^2}}} = 0.102\ \mathrm{s}$$

To find the uncertainty in *t*, we need to use the Error Propagation Formula. For any quantity *F* that is a function of multiple variables *F(x,y,z,…)*, the error in *F* (which we call e_F) is defined as

$$e_F = \sqrt{\left(e_x \frac{\partial F}{\partial x}\right)^2 + \left(e_y \frac{\partial F}{\partial y}\right)^2 + \left(e_z \frac{\partial F}{\partial z}\right)^2 + \cdots}$$

where e_x, e_y, and e_z are the errors in individual measurements. If you are unfamiliar with partial derivatives (such as the term $\frac{\partial F}{\partial x}$), they are simply normal derivatives where all other quantities in the equation are treated as constants *except* the variable with respect to which you are taking the partial derivative.

Let's write out the error propagation formula for the student's experiment. The quantity he wants to calculate is *t*, and *t* is a function of two variables: *t(h,g)* each of which have uncertainties e_h and e_g. So the formula for the error in *t* is:

$$e_t = \sqrt{\left(e_h \frac{\partial t}{\partial h}\right)^2 + \left(e_g \frac{\partial t}{\partial g}\right)^2}$$

Let's evaluate the partial derivatives one by one. To evaluate $\frac{\partial t}{\partial h}$, we would treat the number 2 and the variable *g* as constants, so:

$$\frac{\partial t}{\partial h} = \frac{\partial}{\partial h}\left(\sqrt{\frac{2h}{g}}\right) = \left(\sqrt{\frac{2}{g}}\right)\frac{\partial}{\partial h}\left(h^{1/2}\right) = \left(\sqrt{\frac{2}{g}}\right)\left(\frac{1}{2}h^{-1/2}\right) = \sqrt{\frac{1}{2gh}}$$

To evaluate $\frac{\partial t}{\partial g}$, we would treat the number 2 and the variable *h* as constants, so:

$$\frac{\partial t}{\partial g} = \frac{\partial}{\partial g}\left(\sqrt{\frac{2h}{g}}\right) = \left(\sqrt{2h}\right)\frac{\partial}{\partial g}\left(g^{-1/2}\right) = \left(\sqrt{2h}\right)\left(-\frac{1}{2}g^{-3/2}\right) = -\sqrt{\frac{h}{2g^3}}$$

Now we plug in these expressions to the equation for e_t:

$$e_t = \sqrt{\left(e_h \frac{\partial t}{\partial h}\right)^2 + \left(e_g \frac{\partial t}{\partial g}\right)^2} = \sqrt{\left(e_h \sqrt{\frac{1}{2gh}}\right)^2 + \left(-e_g \sqrt{\frac{h}{2g^3}}\right)^2} = \sqrt{e_h^2 \left(\frac{1}{2gh}\right) + e_g^2 \left(\frac{h}{2g^3}\right)}$$

At this point, all we need to do is plug in numbers for e_h, e_g, h, and g.

$$e_t = \sqrt{\frac{e_h^2}{2gh} + \frac{e_g^2 h}{2g^3}} = \sqrt{\frac{(0.0005\ \mathrm{m})^2}{2\left(9.82\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(0.052\ \mathrm{m})} + \frac{\left(0.04\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)^2 (0.052\ \mathrm{m})}{2\left(9.82\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)^3}} = 0.0005\ \mathrm{s}$$

This means that the final answer the student should report for the amount of time it takes a ball to drop a distance equal to the length of his index finger is 0.102 ± 0.0005 seconds. We can see that is number is very precise, because the uncertainty (0.0005 seconds) is very small.

4.5 Error in Trigonometric Functions

One thing to note is that the error propagation equation assumes that the errors in your variables are normally-distributed (i.e., Gaussian). This means your errors are symmetric: they are equally likely to be a little higher than the true value, or a little lower than the true value. Some quantities do NOT have normally distributed errors (including common trigonometric functions like sine, cosine, and tangent) and thus extra care must be taken when there is a trig function in your expression.

The easiest option is to simply rewrite the trigonometric function based on the sides of the triangle you are measuring, and measure the side lengths, rather than the angle. Thus, you have avoided using trigonometric functions at all, and the errors in the length of the sides of the triangle will be normally distributed.

$$\sin(\theta) = \frac{O}{H}$$
$$\cos(\theta) = \frac{A}{H}$$
$$\tan(\theta) = \frac{O}{A}$$

If you absolutely must measure the angle directly in your experiment and you cannot use trigonometry to measure the sides of a triangle, then you will want to determine the error in the trigonometric function itself by the following equations, and then use this in your error propagation equation.

$$e_{\tan(\theta)} = \left|\frac{\tan(\theta + e_\theta) - \tan(\theta - e_\theta)}{2}\right| \qquad \text{Eq (1)}$$

Here, $e_{\tan(\theta)}$ represents the error in the function $\tan(\theta)$, and the e_θ is the error in θ that you determined during your experiment. The equation above can also be applied to sine and cosine:

$$e_{\sin(\theta)} = \left|\frac{\sin(\theta + e_\theta) - \sin(\theta - e_\theta)}{2}\right| \qquad \text{Eq (2)}$$

$$e_{\cos(\theta)} = \left|\frac{\cos(\theta + e_\theta) - \cos(\theta - e_\theta)}{2}\right| \qquad \text{Eq (3)}$$

Here is an example. If your expression looks like

$$y = \frac{\sin(\theta)}{2x}$$

the best thing to do is replace $\sin(\theta)$ with $\frac{O}{H}$ to obtain the expression

$$y = \frac{O}{2xH}$$

If that is not possible in your particular experiment, then the next best thing to do is replace $\sin(\theta)$ with another variable (let $z = \sin(\theta)$, for example) and rewrite the expression as

$$y = \frac{z}{2x}$$

Then you can find the error in z using Eq. 2 above, and use the error propagation formula to propagate the errors in z and x into an error in your final answer, y.

4.6 Percent Error

After you calculate your final answer and the error on that answer, you should quote a **percent error**. This puts your error value in context, by indicating what percent of your measurement is uncertain. Percent error is defined as

$$\text{Percent Error} = \frac{\text{measurement error}}{\text{measurement}}$$

so in the case described above, the percent error is

$$\text{Percent Error} = \frac{e_t}{t} = \frac{0.0005\ \text{s}}{0.102\ \text{s}} = 0.0053$$

and the student may report that he has a 0.5% error in his calculated time of fall. Note that the student should report both the actual numerical value and error (0.102 ± 0.0005 s) and the percent error (0.5%). If the experiment was one in which a percent difference from the true value can be calculated, the percent difference should be reported as well.

5. Types of Error

A measurement of a physical quantity is always an approximation. The uncertainty in a measurement arises, in general, from three types of errors.

Personal errors - Carelessness, poor technique, or bias on the part of the experimenter. The experimenter may measure incorrectly, or may use poor technique in taking a measurement, or may introduce a bias into measurements by expecting (and inadvertently forcing) the results to agree with the expected outcome. Gross personal errors, sometimes called mistakes or blunders, should be avoided and corrected if discovered. As a rule, gross personal errors are excluded from the error analysis discussion because it is assumed that the experimental result was obtained by following correct procedures. The term "human error" should also be avoided in error analysis discussions because it is too vague to be useful.

Systematic errors: These are errors which can be traced to an imperfectly made instrument, a badly calibrated instrument, or to the personal technique and bias of the observer. Systematic errors consistently skew measured values in the same direction (too high or too low). Systematic errors cannot be detected or reduced by increasing the number of observations, and can be reduced by applying a correction or correction factor to compensate for the effect.

Random errors: These are errors for which the causes are unknown or indeterminate, but are usually small and follow the laws of chance. Random errors can be reduced by averaging over a large number of observations, as described in Section 4.2.

The following are some examples of systematic and random errors to consider when writing your error analysis.

- **Incomplete definition** (may be systematic or random) - One reason that it is impossible to make exact measurements is that the measurement is not always clearly defined. For example, if two different people measure the length of the same rope, they would probably get different results because each person may stretch the rope with a different tension. The best way to minimize definition errors is to carefully consider and specify the conditions that could affect the measurement.

- **Failure to account for a factor** (usually systematic) – The most challenging part of designing an experiment is trying to control or account for all possible factors except the one independent variable that is being analyzed. For instance, you may inadvertently ignore air resistance when measuring free-fall acceleration, or you may fail to account for the effect of the Earth's magnetic field when measuring the field of a small magnet. The best way to account for these sources of error is to brainstorm with your peers about all the factors that could possibly affect your result. This brainstorm should be done before beginning the experiment so that arrangements can be made to account for the confounding factors before taking data. Sometimes a correction can be applied to a result after taking data to account for an error that was not detected.

- **Environmental factors** (systematic or random) - Be aware of errors introduced by your immediate working environment. You may need to take account for or protect your experiment from vibrations, drafts, changes in temperature, electronic noise or other effects from nearby apparatus.

- **Instrument resolution** (random) - All instruments have finite precision that limits the ability to resolve small measurement differences. For instance, a meter stick cannot distinguish distances to a precision much better than about half of its smallest scale division (0.5 mm in this case).

- **Failure to calibrate or check zero of instrument** (systematic) - Whenever possible, the calibration of an instrument should be checked before taking data. If a calibration standard is not available, the accuracy of the instrument should be checked by comparing with another instrument that is at least as precise, or by consulting the technical data provided by the manufacturer. When making a measurement with an electronic device, always check the zero reading first. Re-zero the instrument if possible, or measure the displacement of the zero reading from the true zero and correct any measurements accordingly. It is a good idea to check the zero reading throughout the experiment.

- **Parallax** (systematic or random) - This error can occur whenever there is some distance between the measuring scale and the indicator used to obtain a measurement. If the observer's eye is not squarely aligned with the pointer and scale, the reading may be too high or low (some analog meters have mirrors to help with this alignment).

- **Instrument drift** (systematic) - Most electronic instruments have readings that drift over time. The amount of drift is generally not a concern, but occasionally this source of error can be significant and should be considered.

- **Physical variations** (random) - It is always wise to obtain multiple measurements over the entire range being investigated. Doing so often reveals variations that might otherwise go undetected. These variations may call for closer examination, or they may be combined to find an average value.

- **Lag time** (systematic) - Some measuring devices require time to reach equilibrium, and taking a measurement before the instrument is stable will result in a measurement that is generally too low. The most common example is taking temperature readings with a thermometer that has not reached thermal equilibrium with its environment.