The background of the slide features several horizontal staves of musical notation. The notes and symbols are rendered in a light, semi-transparent grey, creating a subtle pattern across the entire page. The notation includes various note values, stems, and rests, typical of a musical score.

Mathematics and Music: *Singing the Same Tune*

Michael Peterson
Spring, 2022

Discipline Organization at Las Positas College

STEM

- Mathematics
- Physics
- Chemistry
- Computer Science
- Biology
- Others!

Arts and Humanities

- Music
- Art
- Dance
- English
- Foreign Languages
- Others!

Discipline Organization in the Middle Ages

Trivium

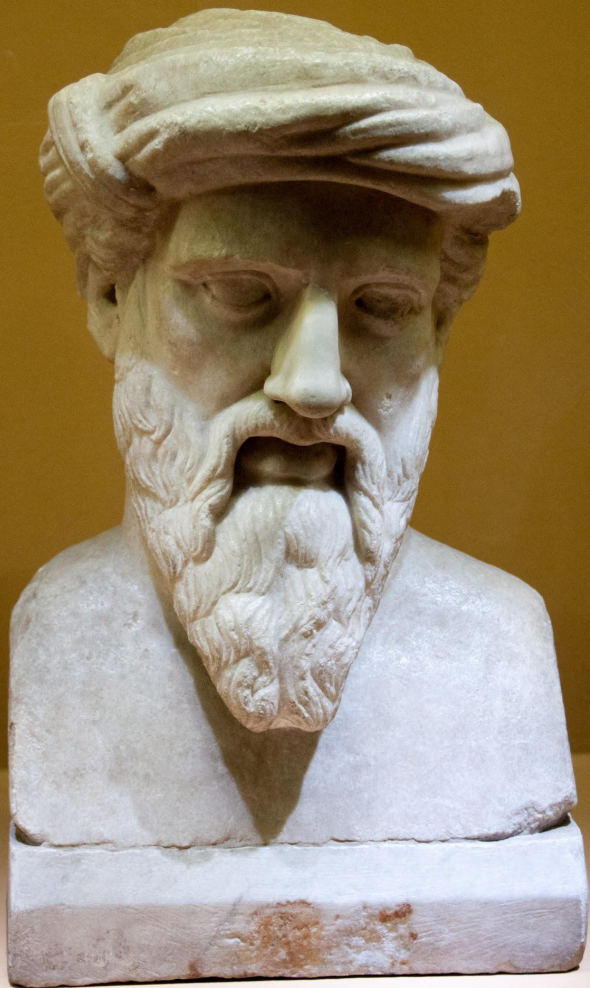
("Arts of the Word")

- Grammar
- Logic
- Rhetoric

Quadrivium

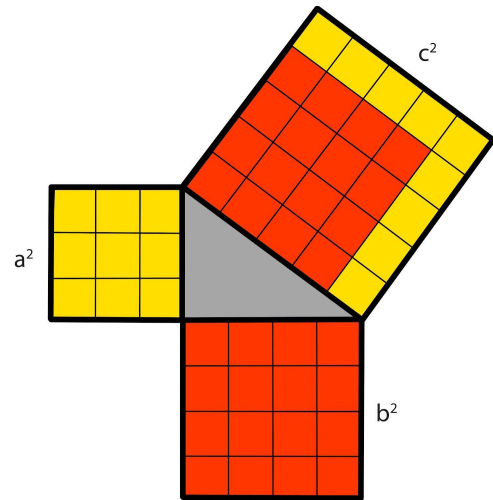
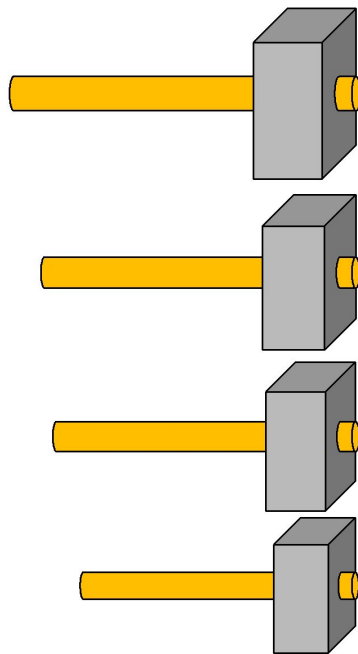
("Arts of the Number")

- Arithmetic
- Geometry
- Astronomy
- Music



Pythagoras of Samos

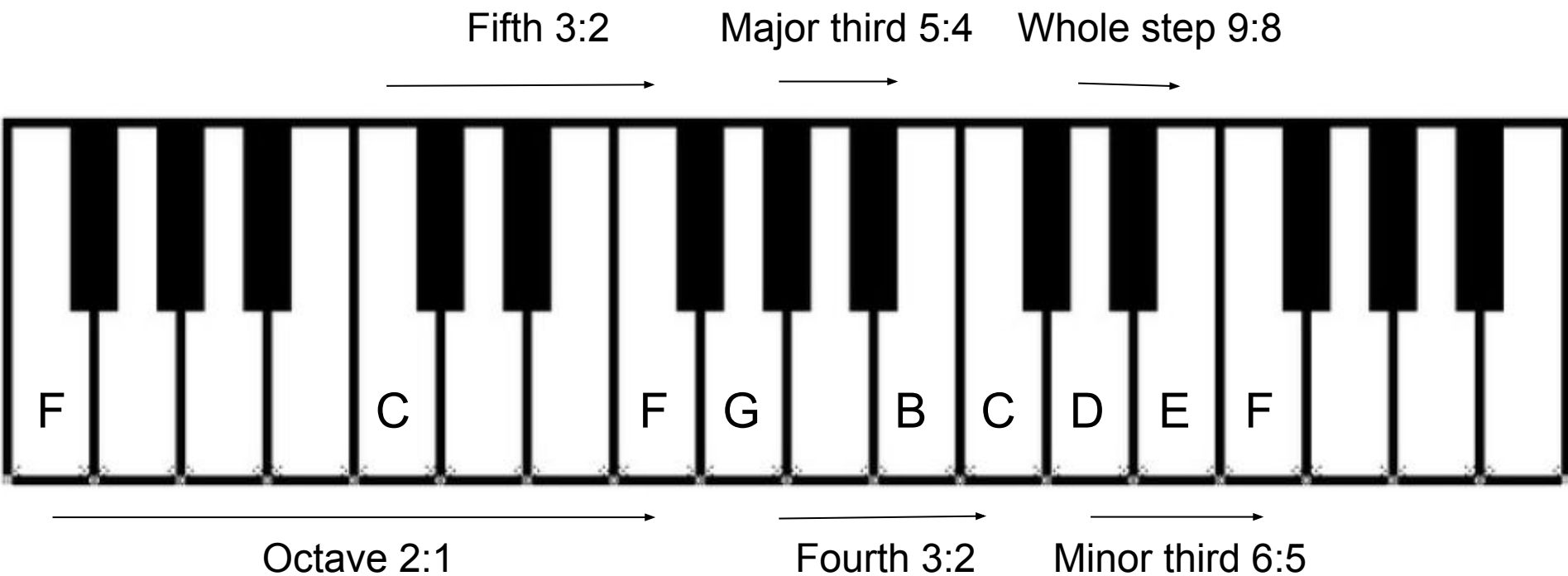
c. 570 - c. 495 BCE



$$a^2 + b^2 = c^2$$



Every music interval corresponds with a ratio.



**Leonhard Euler classified intervals by dissonance/consonance
(i.e. “order of softness”)**

Interval	Ratio	Order of Softness
Unison	1:1	1
Octave	1:2	2
Octave + Fifth	1:3	3
Two octaves	1:4	3
Three octaves	1:8	4
Two octaves + Major third	1:5	5

Euler's Gradus Function

Let n be the product of the two numbers in the interval's ratio.
Suppose its prime factorization is

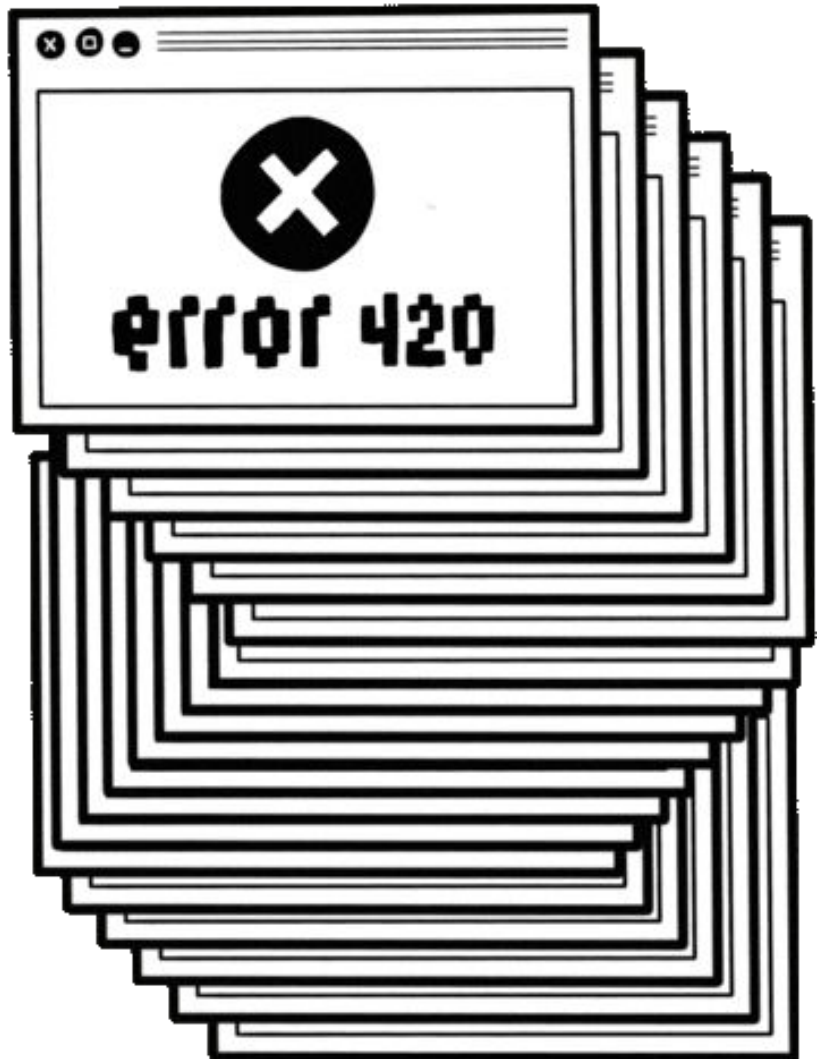
$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \quad (p_i \text{ distinct primes, } a_i \geq 1)$$

Then the order of softness is

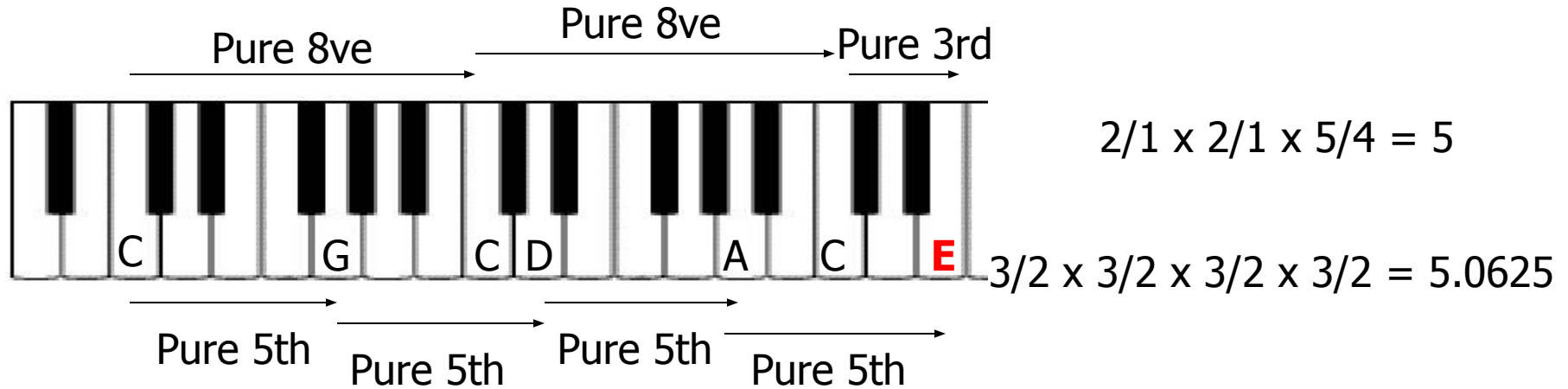
$$E(n) = 1 + \sum_{k=1}^r a_k (p_k - 1)$$

But ...

On a keyboard, not
possible for every interval
to have these ratios

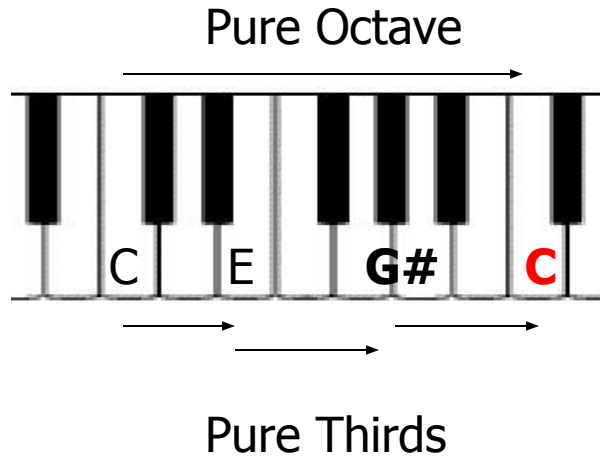


Tuning from C to E Differently



Results in Different E's!

Tuning Octaves using Thirds



$$5/4 \times 5/4 \times 5/4 = 1.953125 \neq 2$$

Gives Out of Tune Octaves!

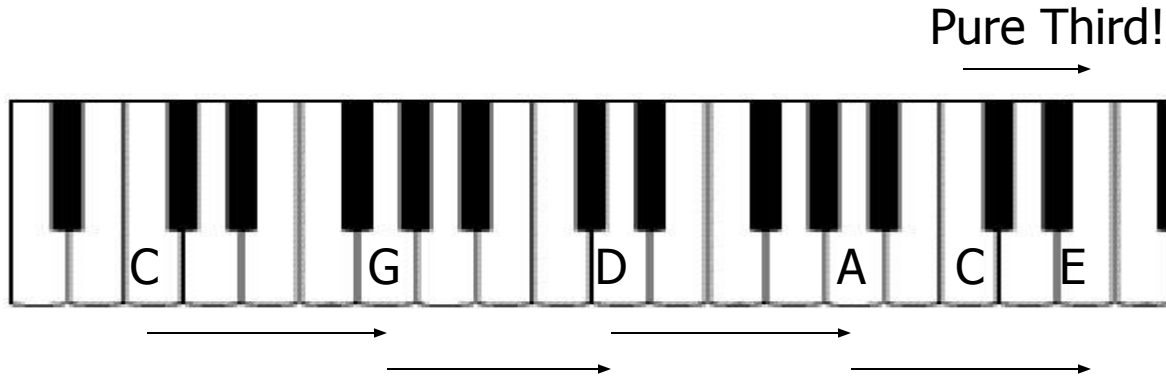


What do we do?

Pick an interval to be in tune

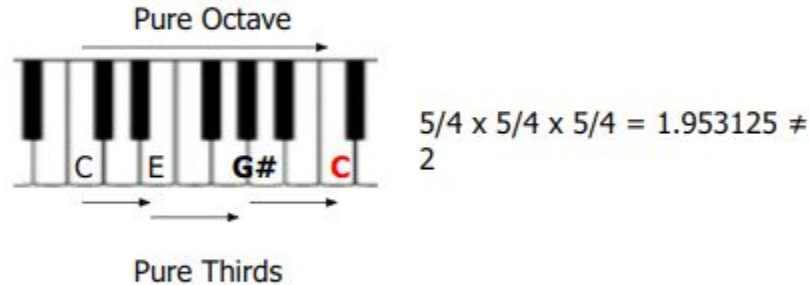
Quarter-Comma Meantone

Common in 16th and 17th Centuries



Slightly Narrowed (Out of tune) Fifths

But ... Tuning Octaves using Thirds



Gives Out of Tune Octaves!

As you tune up ... you get more sharps

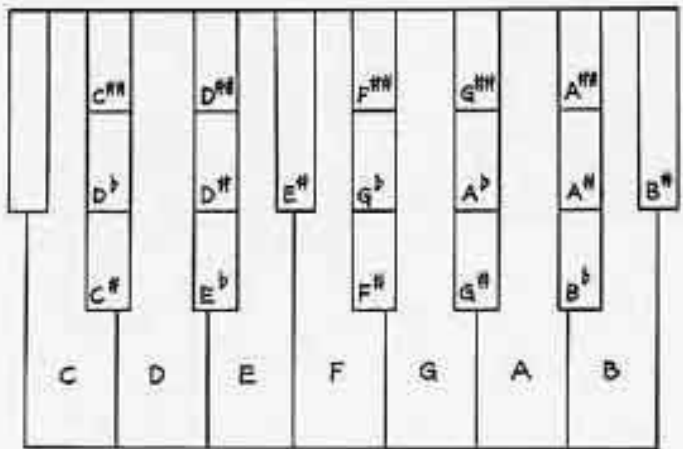
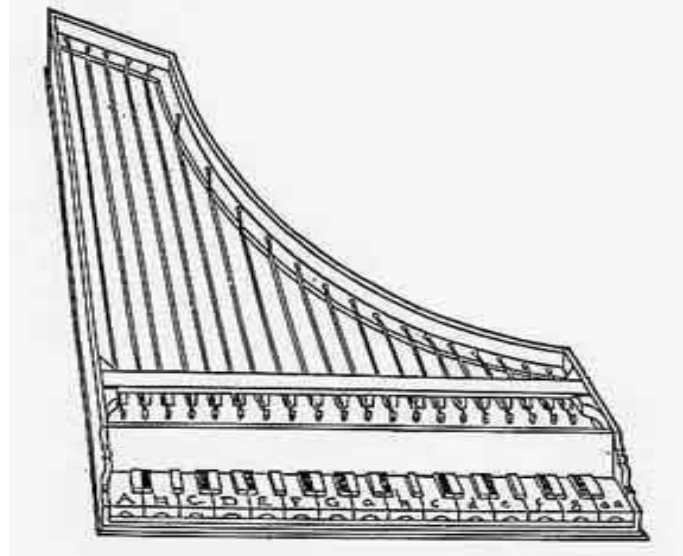
As you tune down ... you get more flats

Sharps \neq Flats

Chromatic Harpsichord



Chromatic Harpsichord Illustrations by Giozeffo Zarlino



The Problem with Chromatic Harpsichords

Impractical to Play, Build or Maintain

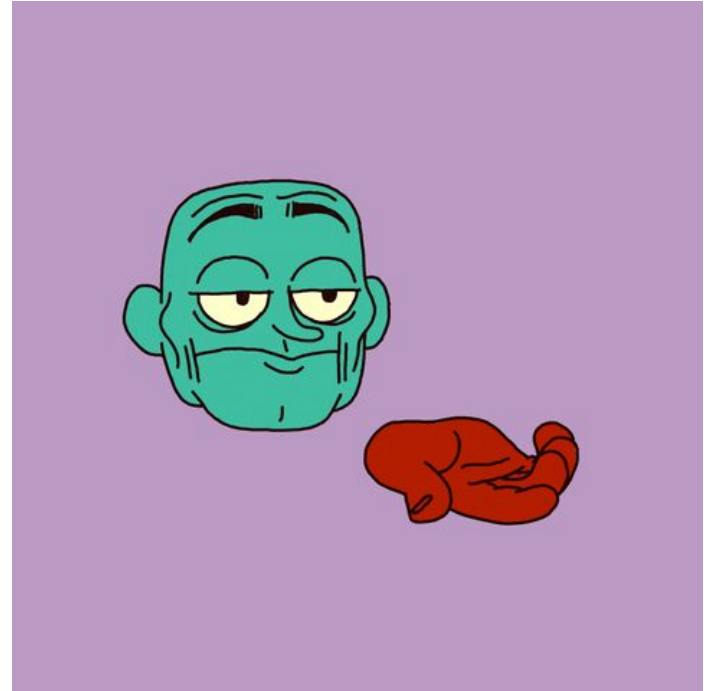
I JUST...

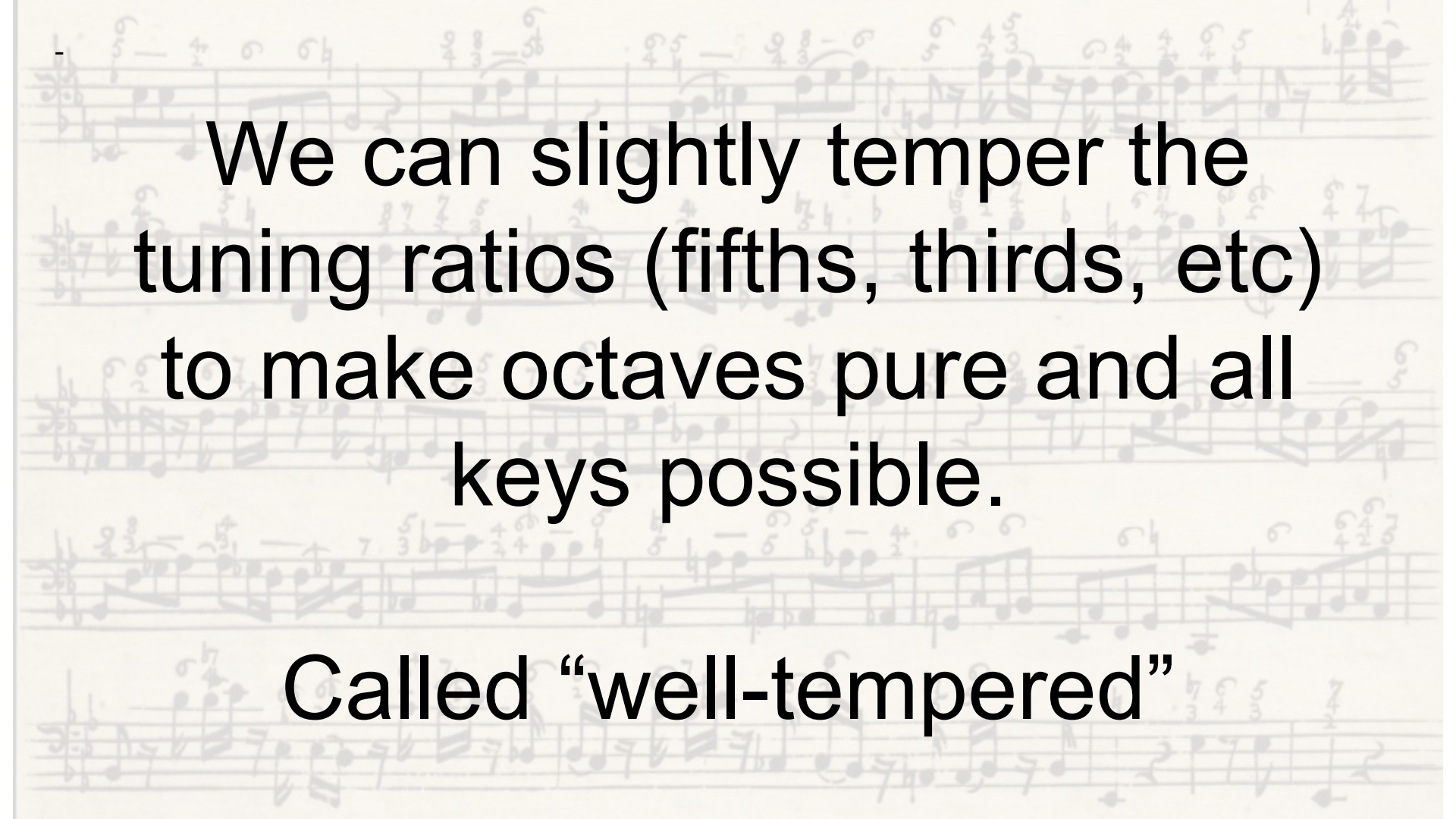


What do we do?

Pick another interval to
be in tune

Hmmm ... what about
octaves?



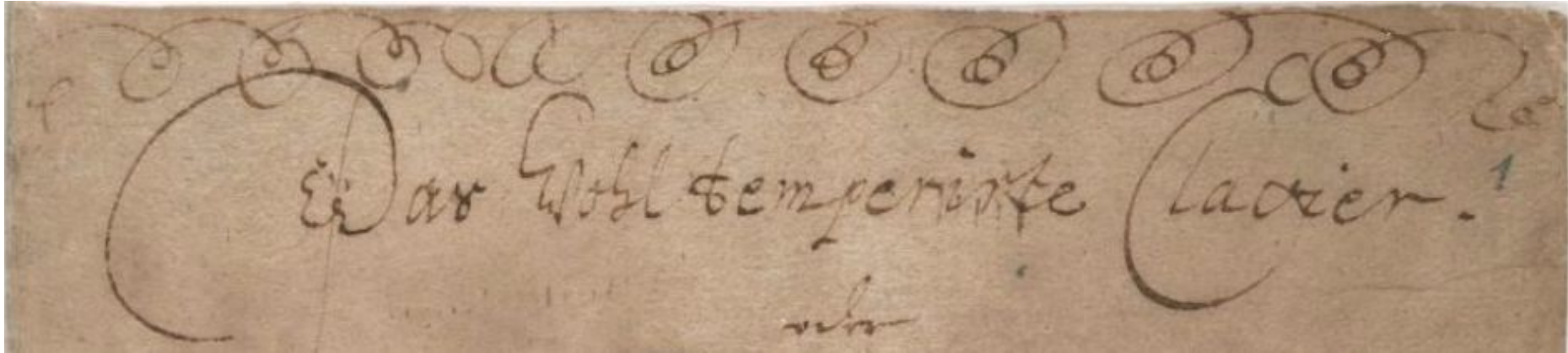
The background of the slide is a light gray musical score with various notes, rests, and clefs on a grid of staves. The text is overlaid on this background.

We can slightly temper the tuning ratios (fifths, thirds, etc) to make octaves pure and all keys possible.

Called “well-tempered”

The Well-Tempered Clavier (1722) Johann Sebastian Bach

First collection in all possible keys (major and minor) because of this new way of tuning



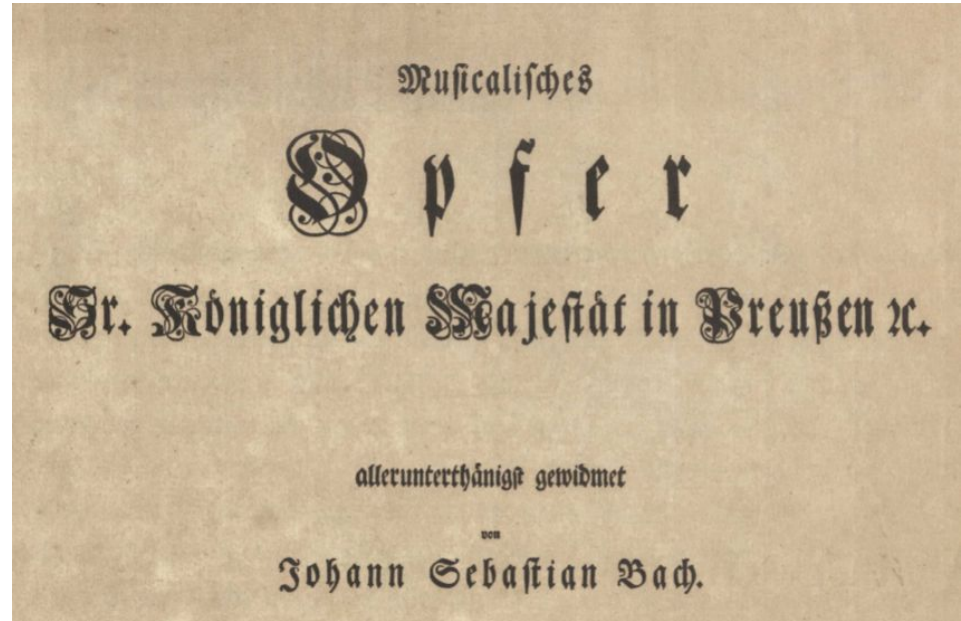
1747: Bach joined the Mizler Society for Correspondence in the Musical Sciences

This society was founded to:

- Establish a musical science based on mathematics, philosophy, and connection to nature
- Share writings/papers with each other



Bach's first Submission to the Mizler Society: The Musical Offering



Canon I a 2. *Canones diversi super Thema Regium*

Handwritten musical score for Canon I a 2. The score consists of two staves of music. The top staff begins with a treble clef, a key signature of two flats (B-flat and E-flat), and a common time signature (C). The bottom staff begins with a bass clef, the same key signature, and a common time signature. The music is written in a cursive, handwritten style. The bottom staff concludes with the instruction "2. a 2 Violin: in Unisono." written in a similar cursive hand.

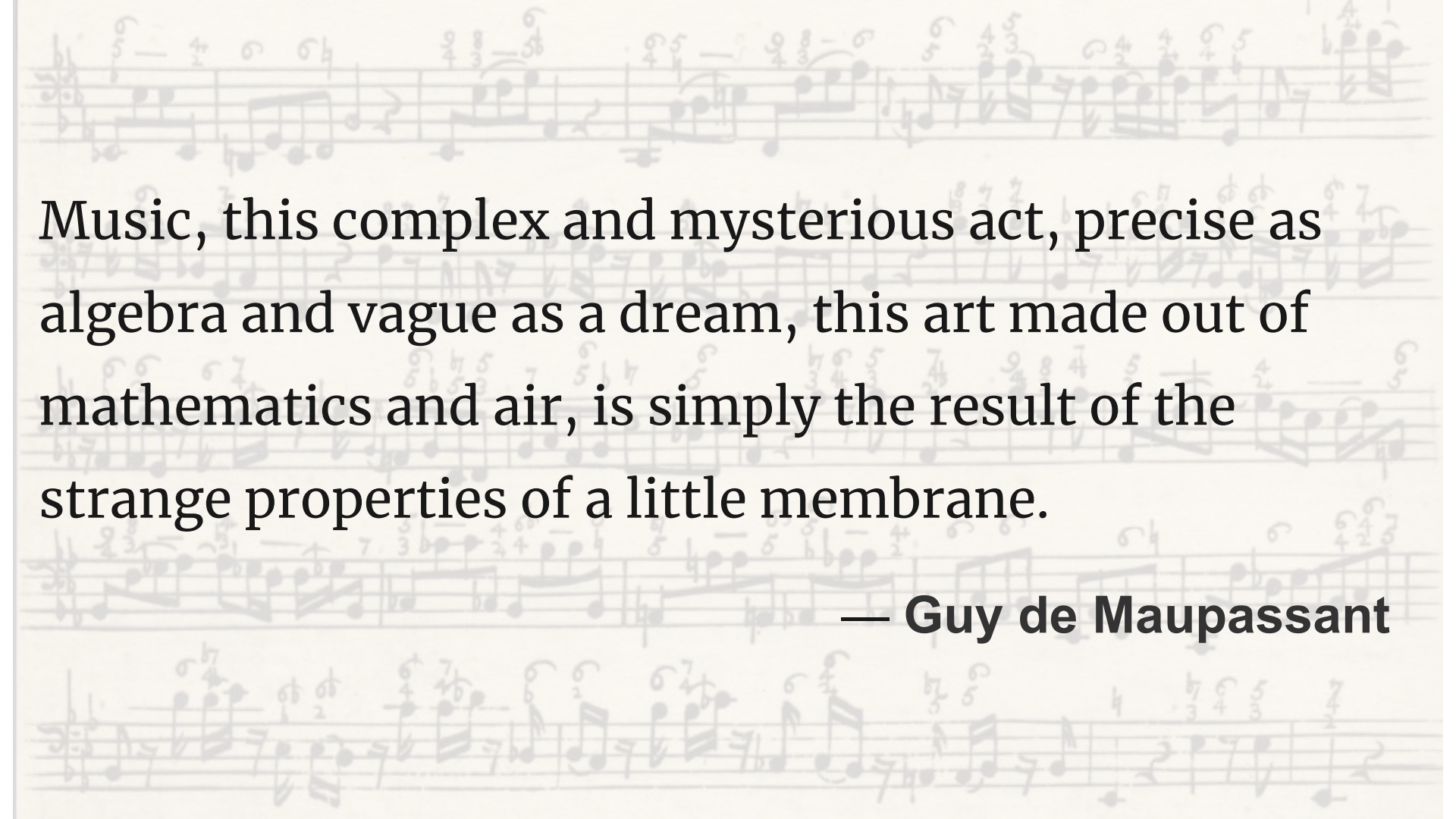
Canon a 2.

1.

Printed musical score for Canon a 2. The score consists of three staves of music. The top staff begins with a treble clef, a key signature of two flats (B-flat and E-flat), and a common time signature (C). The middle and bottom staves begin with a bass clef, the same key signature, and a common time signature. The music is written in a clear, printed style. The bottom staff concludes with a double bar line and a common time signature.



Johann Sebastian Bach.

The background of the image consists of several horizontal staves of musical notation. The notes, clefs, and other symbols are rendered in a light, semi-transparent grey color, creating a subtle, artistic backdrop for the text.

Music, this complex and mysterious act, precise as algebra and vague as a dream, this art made out of mathematics and air, is simply the result of the strange properties of a little membrane.

— **Guy de Maupassant**

5. a 2.

This image shows a handwritten musical score on aged paper. It consists of two systems of staves. The first system has a treble clef on the top staff and a bass clef on the bottom staff. The second system also has a treble clef on the top staff and a bass clef on the bottom staff. The music is written in a style typical of 18th or 19th-century manuscripts, with various note values, rests, and accidentals. The paper shows signs of age, including some staining and discoloration.

5. a 2.
Thema.

This image shows a printed musical score for a piano piece. It consists of two systems of staves. The first system has a treble clef on the top staff and a bass clef on the bottom staff. The second system also has a treble clef on the top staff and a bass clef on the bottom staff. The music is written in a style typical of 18th or 19th-century manuscripts, with various note values, rests, and accidentals. The paper shows signs of age, including some staining and discoloration.

This image shows a printed musical score for a piano piece. It consists of two systems of staves. The first system has a treble clef on the top staff and a bass clef on the bottom staff. The second system also has a treble clef on the top staff and a bass clef on the bottom staff. The music is written in a style typical of 18th or 19th-century manuscripts, with various note values, rests, and accidentals. The paper shows signs of age, including some staining and discoloration.

1st Time: C Minor (back where we started!)

The first system of music is written in C minor, 3/4 time. It consists of three staves: a treble clef staff, a treble clef staff, and a bass clef staff. The key signature has three flats (Bb, Eb, Ab). The first staff contains a melody with quarter and eighth notes. The second staff contains a melody with eighth and sixteenth notes. The third staff contains a bass line with eighth and sixteenth notes. A vertical yellow highlight is present in the middle of the system, covering the second measure of all three staves.

The second system of music is also in C minor, 3/4 time, and consists of three staves: a treble clef staff, a treble clef staff, and a bass clef staff. The key signature has three flats (Bb, Eb, Ab). The first staff contains a melody with quarter and eighth notes. The second staff contains a melody with eighth and sixteenth notes. The third staff contains a bass line with eighth and sixteenth notes. An annotation "→ C minor" is written at the end of the first staff in the second system.