Mathematics and Music:
Singing the Same Tune

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Discipline Organization at Las Positas College

STEM

- Mathematics
- Physics
- Chemistry
- Computer Science
- Biology
- Others!

Arts and Humanities

- Music
- Art
- Dance
- English
- Foreign Languages
- Others!

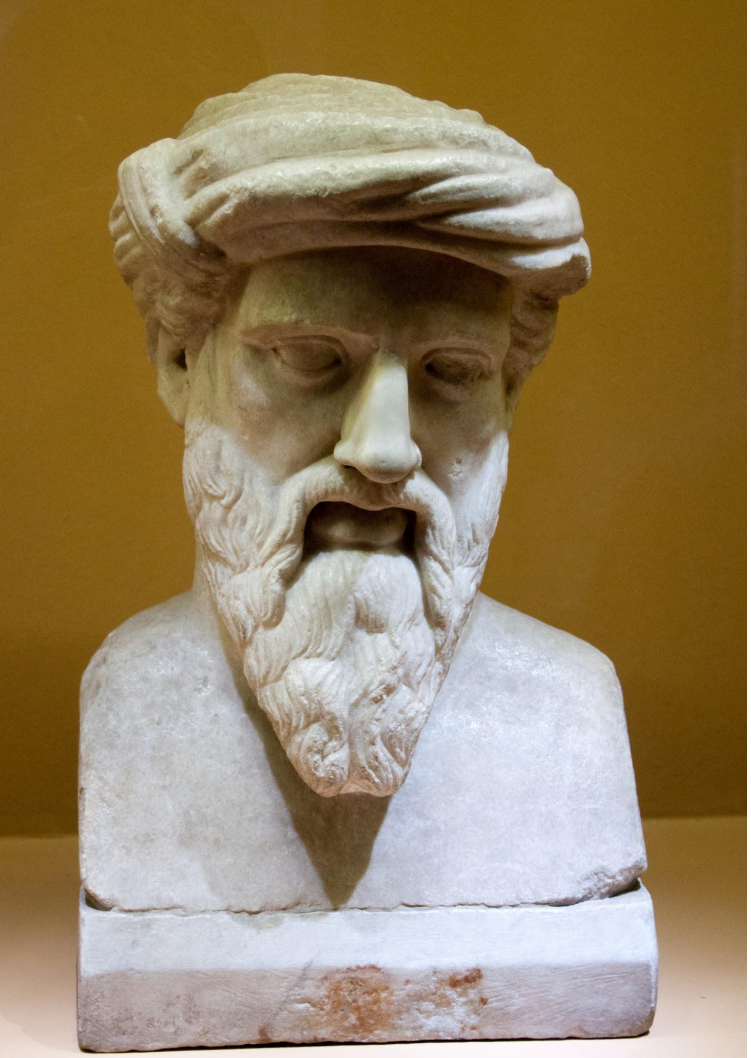
Discipline Organization in the Middle Ages

Trivium
("Arts of the Word")

- Grammar
- Logic
- Rhetoric

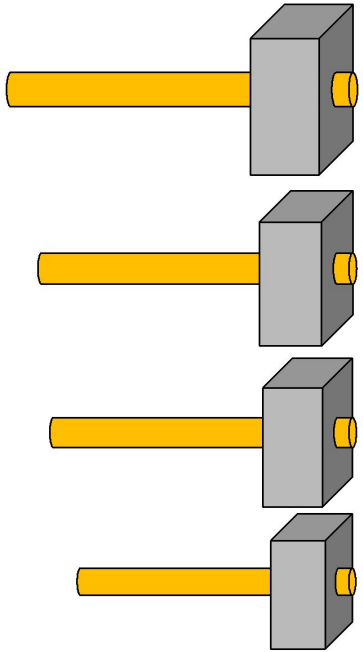
Quadrivium
("Arts of the Number")

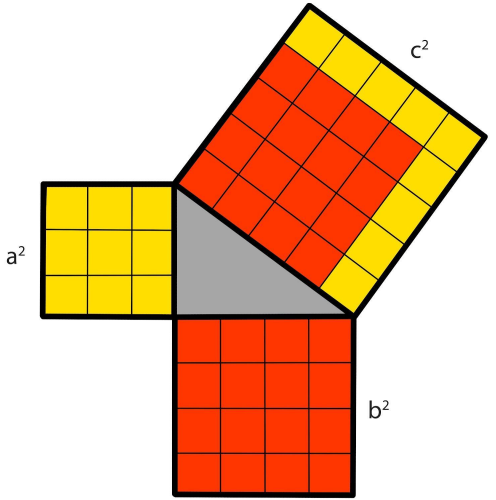
- Arithmetic
- Geometry
- Astronomy
- Music



Pythagoras of Samos

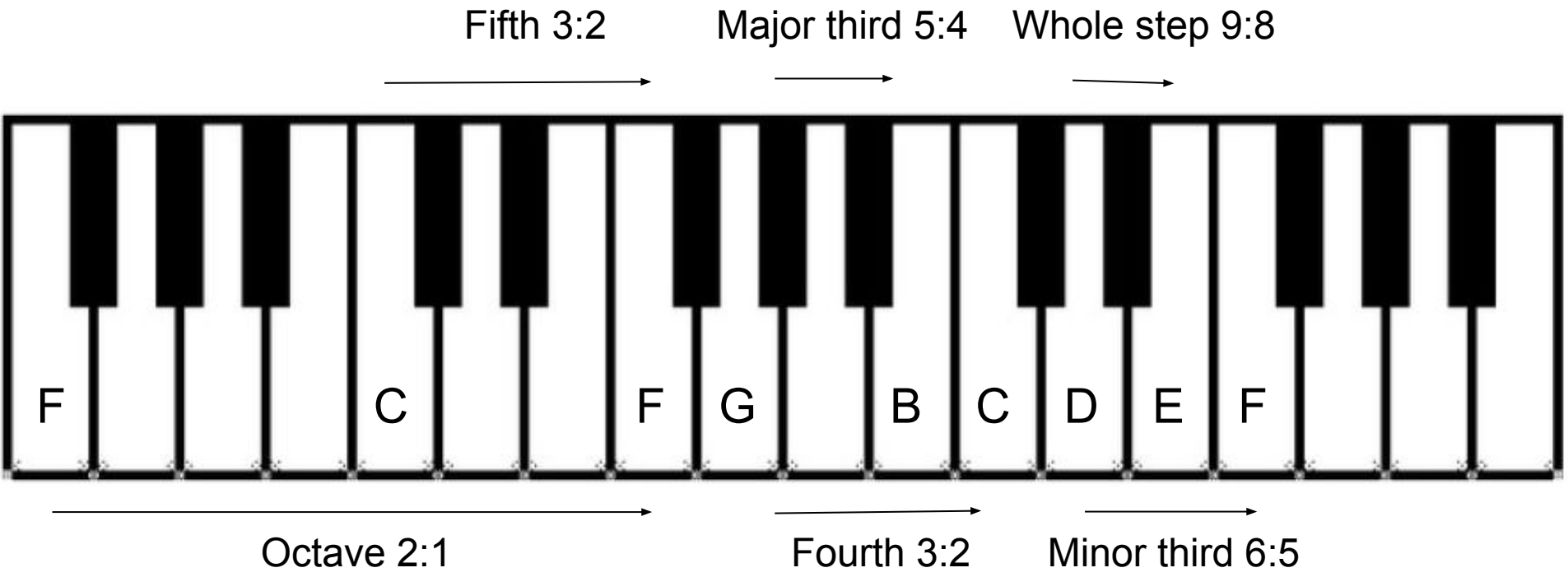
c. 570 - c. 495 BCE





$$a^2 + b^2 = c^2$$

Every music interval corresponds with a ratio.



Leonhard Euler classified intervals by dissonance/consonance (i.e. “order of softness”)

Interval	Ratio	Order of Softness
Unison	1:1	1
Octave	1:2	2
Octave + Fifth	1:3	3
Two octaves	1:4	3
Three octaves	1:8	4
Two octaves + Major third	1:5	5

Euler's Gradus Function

Let n be the product of the two numbers in the interval's ratio. Suppose its prime factorization is

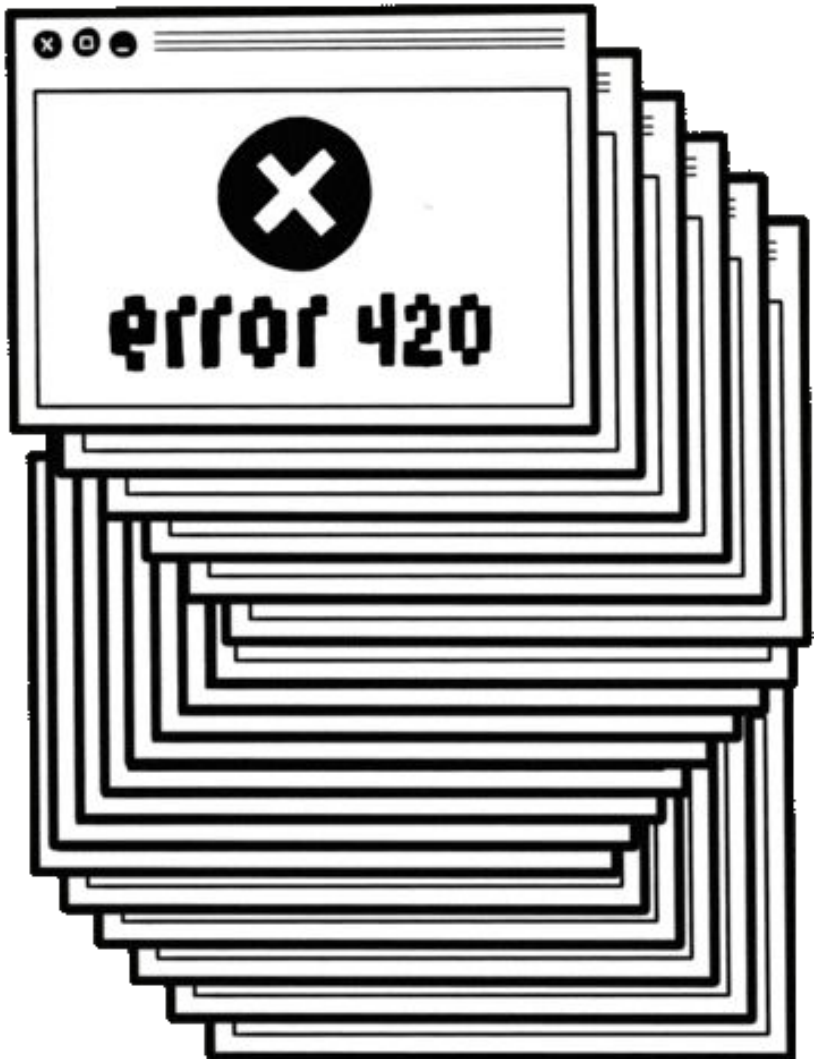
$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \ (p_i \text{ distinct primes}, a_i \geq 1)$$

Then the order of softness is

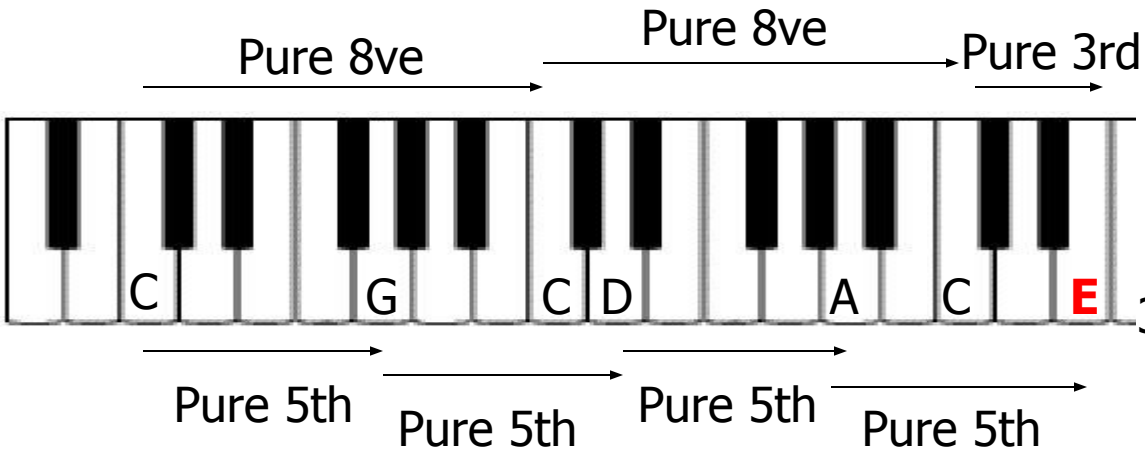
$$E(n) = 1 + \sum_{k=1}^{r} a_k (p_k - 1)$$

But …

On a keyboard, not possible for every interval to have these ratios



Tuning from C to E Differently

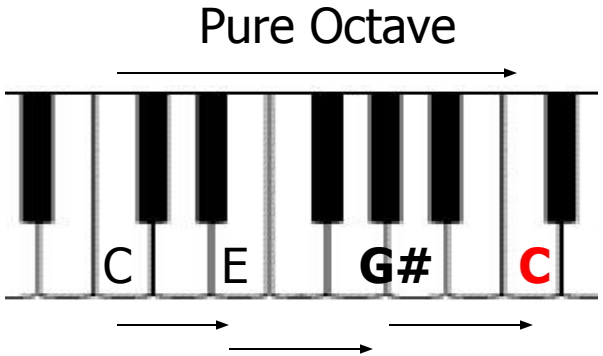


2/1 x 2/1 x 5/4 = 5

3/2 x 3/2 x 3/2 x 3/2 = 5.0625

Results in Different E's!

Tuning Octaves using Thirds



$5/4 \times 5/4 \times 5/4 = 1.953125 \neq 2$

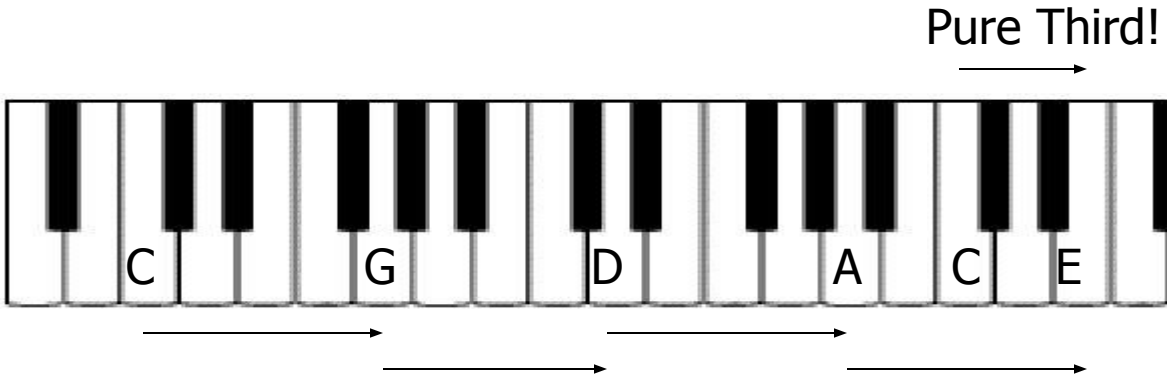
Gives Out of Tune Octaves!

What do we do?

Pick an interval to be in tune

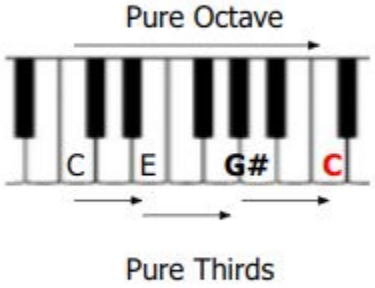
Quarter-Comma Meantone

Common in 16th and 17th Centuries



Slightly Narrowed (Out of tune) Fifths

But … Tuning Octaves using Thirds



5/4 x 5/4 x 5/4 = 1.953125 ≠ 2

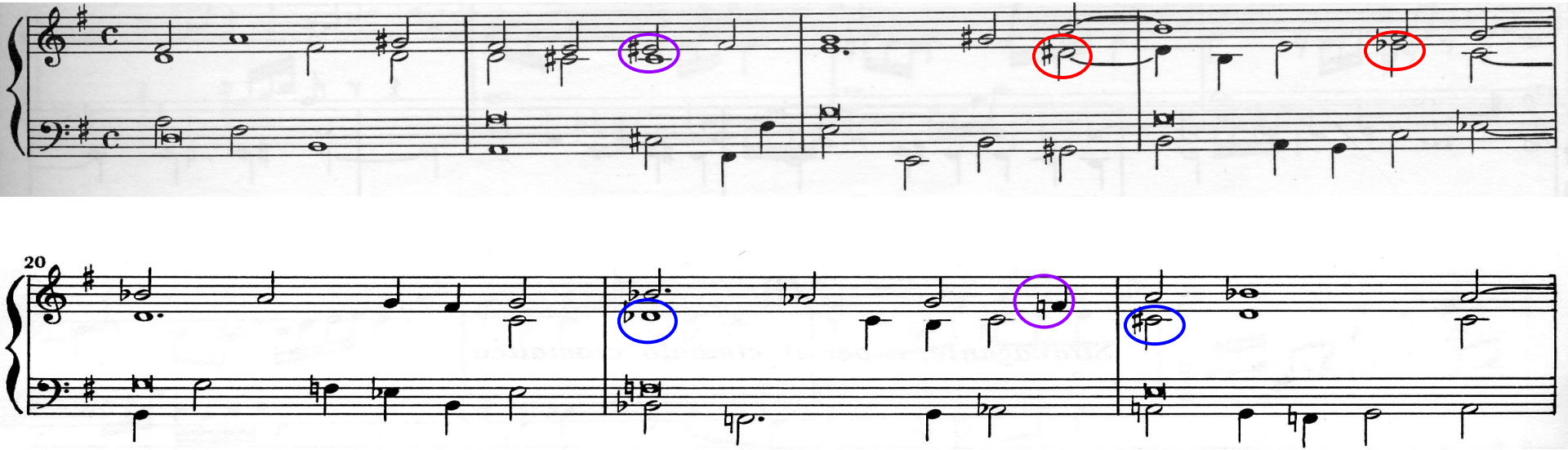
Gives Out of Tune Octaves!

As you tune up … you get more sharps

As you tune down … you get more flats

Sharps ≠ Flats

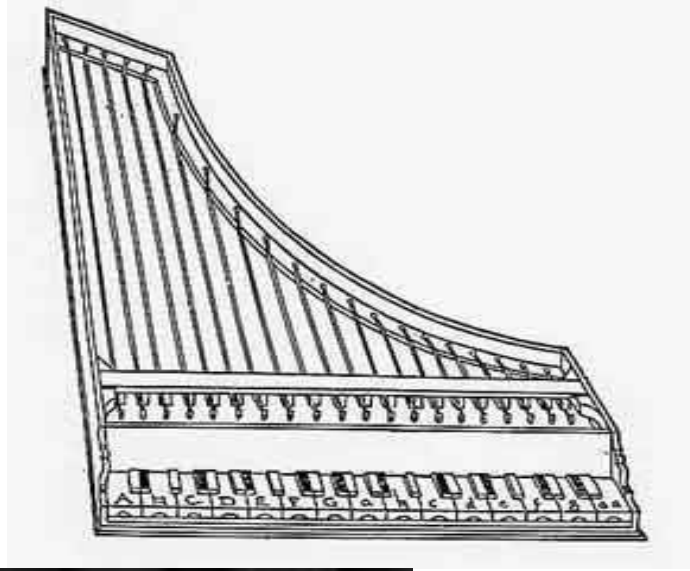
Passages that Do Not Work in Quarter-Comma Meantone

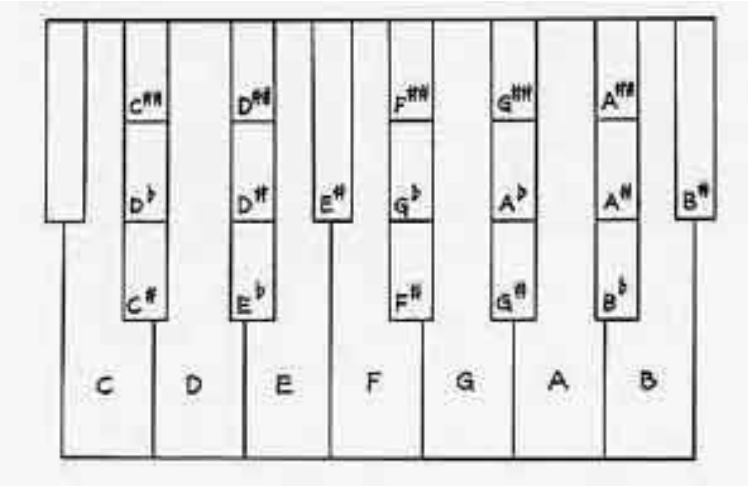


Excerpts of Sonata VII from *Sonate di Cimbalo in varie maniere sopra l'Ave Maris Stella*
by Gioan Pietro del Buono (d. 1657)

Chromatic Harpsichord

Chromatic Harpsichord Illustrations by Giozeffo Zarlino





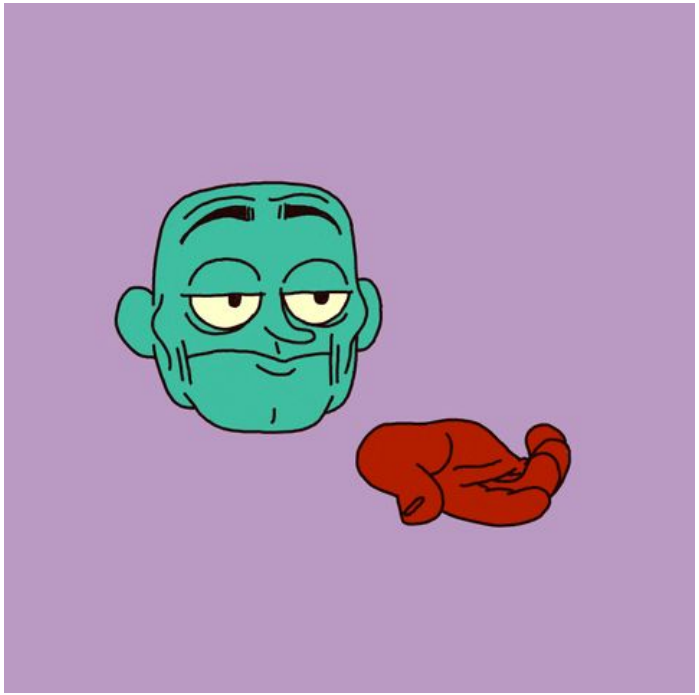
The Problem with Chromatic Harpsichords

Impractical to Play, Build or Maintain

What do we do?

Pick another interval to be in tune

Hmmm … what about octaves?

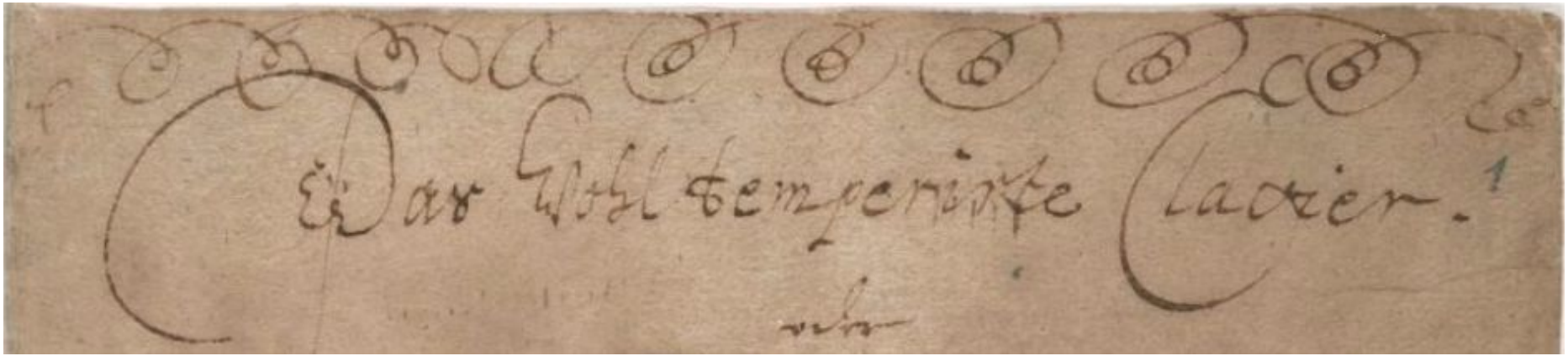


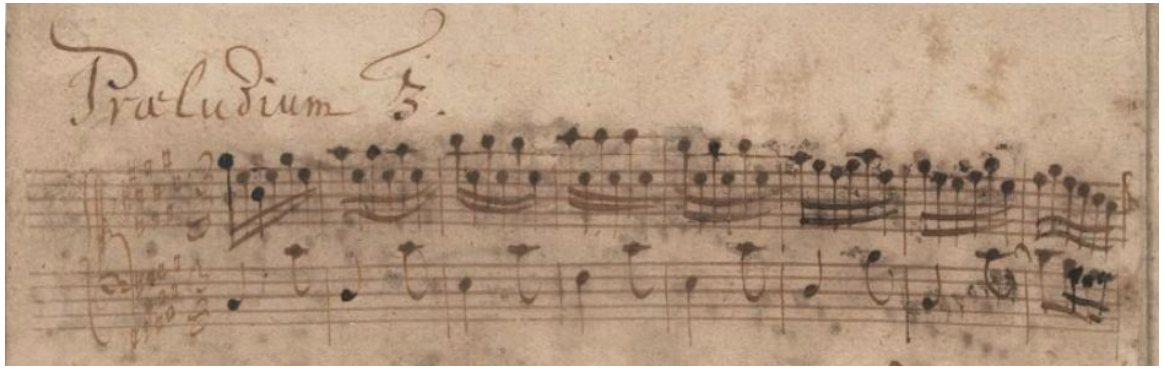
We can slightly temper the tuning ratios (fifths, thirds, etc) to make octaves pure and all keys possible.

Called “well-tempered”

The Well-Tempered Clavier (1722)
Johann Sebastian Bach

First collection in all possible keys (major and minor) because of this new way of tuning



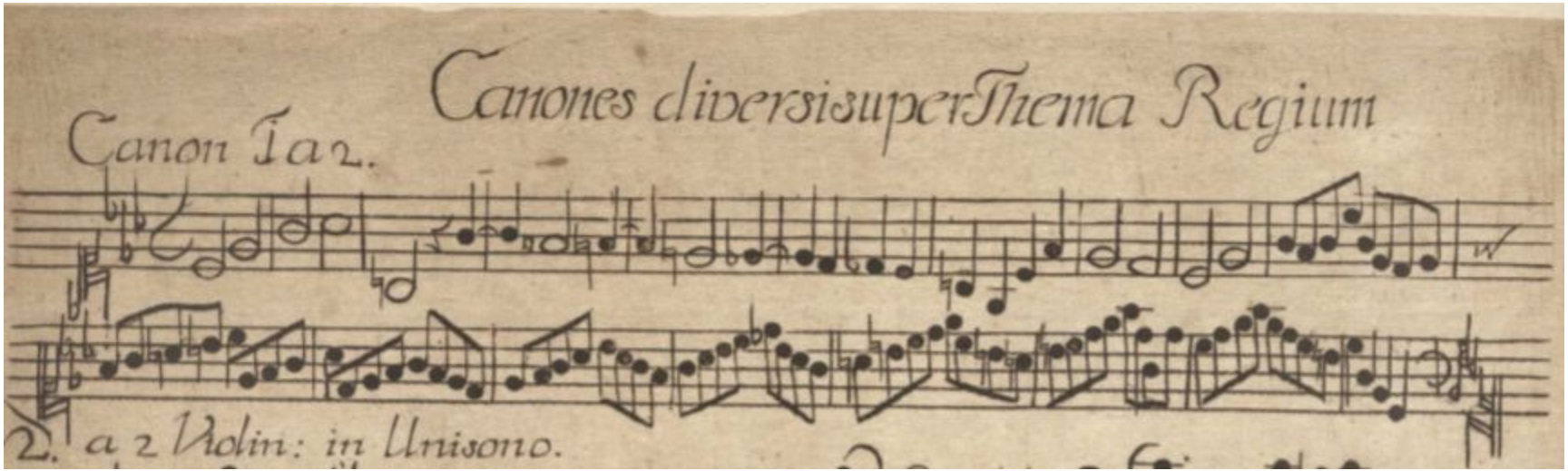


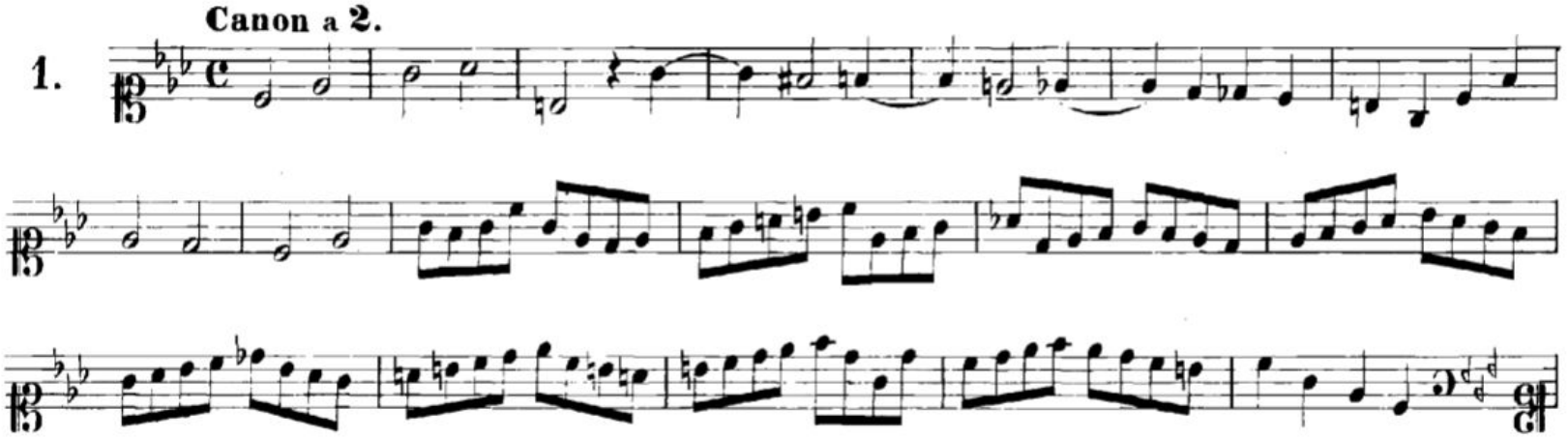
1747: Bach joined the Mizler Society for Correspondence in the Musical Sciences

This society was founded to:

- Establish a musical science based on mathematics, philosophy, and connection to nature
- Share writings/papers with each other

Bach's first Submission to the Mizler Society: The Musical Offering


Canones diversi super Thema Regium
Canon 1a 2.
2. a 2 Violin: in Unisono.

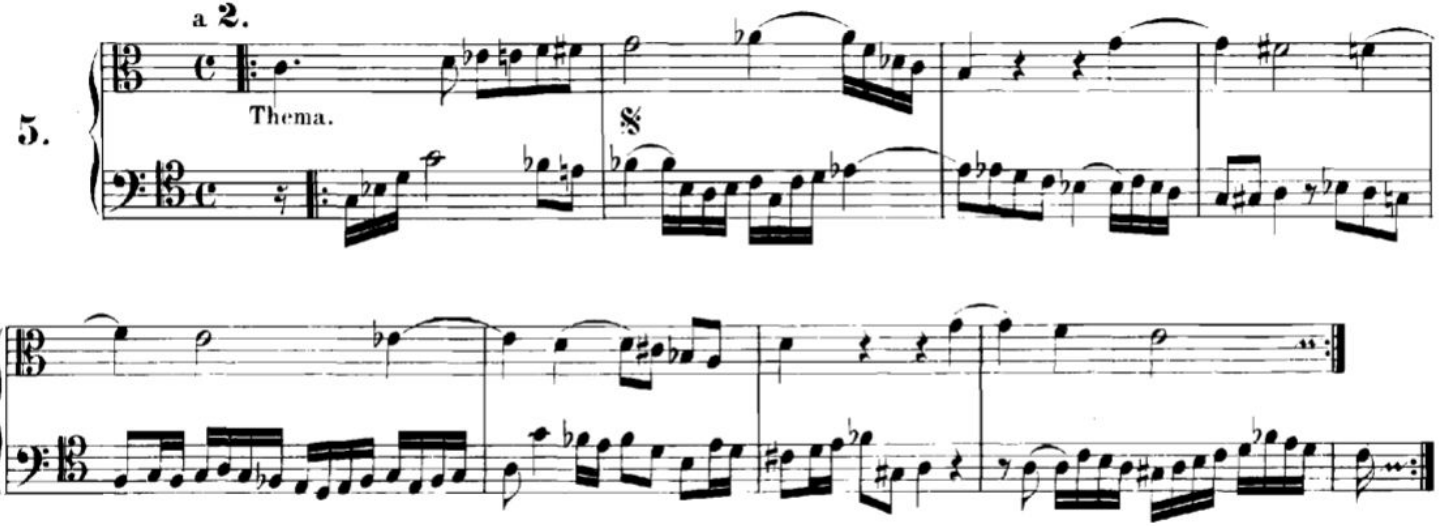

Canon a 2.
1.


Johann Sebastian Bach.

Music, this complex and mysterious act, precise as algebra and vague as a dream, this art made out of mathematics and air, is simply the result of the strange properties of a little membrane.

— **Guy de Maupassant**

5. a 2.



1st Time: C Minor (back where we started!)